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A robust semi-explicit difference scheme for the Kuramoto–Tsuzuki equation

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ABSTRACT

In this paper, we propose a robust semi-explicit difference scheme for solving the Kuramoto–Tsuzuki equation with homogeneous boundary conditions. Because the prior estimate in L_∞-norm of the numerical solutions is very hard to obtain directly, the proofs of convergence and stability are difficult for the difference scheme. In this paper, we first prove the second-order convergence in L_2-norm of the difference scheme by an induction argument, then obtain the estimate in L_∞-norm of the numerical solutions. Furthermore, based on the estimate in L_∞-norm, we prove that the scheme is also convergent with second order in L_∞-norm. Numerical examples verify the correction of the theoretical analysis.

1. Introduction

In this paper, we consider the Kuramoto–Tsuzuki equation

$$\frac{\partial w}{\partial t} = (1+\mathrm{i}c_1)\frac{\partial^2 w}{\partial x^2} + w - (1+\mathrm{i}c_2)|w|^2 w, \quad (x,t) \in (0,1) \times (0,T], \tag{1.1}$$

with the initial condition

$$w(x,0) = w_0(x), \quad x \in [0,1], \tag{1.2}$$

and homogeneous boundary conditions

$$\frac{\partial w}{\partial x}(0,t) = 0, \quad \frac{\partial w}{\partial x}(1,t) = 0, \quad t \in (0,T], \tag{1.3}$$

where c_1 and c_2 are two real constants, $w(x,t)$ is an unknown complex function and $w_0(x)$ is a given complex function.

Tsertsvadze [1] constructed a nonlinear implicit scheme for the 1D Kuramoto–Tsuzuki equation and proved that the scheme is convergent at the rate of order $O(h^{3/2})$ in the discrete L_2-norm and the rate of $O(h)$ in the uniform norm under the requirement $\tau = O(h^{2+\varepsilon})(\varepsilon > 0)$. Sun [2] proved that the scheme in [1] is convergent at the rate of order $O(h^2+\tau^2)$ in the uniform norm without any restrictions on the mesh size; Sun [3–5], Ivanauskas [6,7], Abidi et al. [8] and Omrani [9,10] also constructed and studied some other difference schemes and finite element schemes for the 1D Kuramoto–Tsuzuki equation. Proofs of convergence of all the schemes in [1–10] are based on the L_∞-norm prior estimate of numerical solutions, but it is difficult to obtain the L_∞-norm prior estimate of the numerical solution of our scheme. So the classical method cannot be used in proving the convergence of the new scheme. It should be pointed out that the induction argument is very useful

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for proving convergence of a difference scheme whose prior estimate is difficult to obtain; so it is used widely in many papers [11–15]. In this paper, we propose a new difference scheme which is linearized, and prove that the new scheme is convergent with second order in the uniform norm by the induction argument. We also will give two numerical examples to verify the correction of our theoretical analysis.

The remainder of this paper is arranged as follows. A new difference scheme is given in Section 2. In Section 3, the convergence and stability for the new scheme are proved, and another similar scheme is proposed. In Section 4, we discuss difference schemes for solving inhomogeneous equation. Section 5 is devoted to the numerical tests of the new schemes and shows the correction of the theoretical analysis.

2. Finite difference schemes

We consider the finite difference method for the problem (1.1)–(1.3). As usual, the following notations are used:

$$x_j = jh, \quad 0 \le j \le J = \left[\frac{1}{h}\right], \quad t_n = n\tau, \quad n = 0, 1, 2, \dots, \left[\frac{T}{\tau}\right],$$

$$u_j^n = u(x_j, t_n), \quad U_j^n \approx u(x_j, t_n), \quad u_j^{n+\frac{1}{2}} = \frac{u_j^{n+1} + u_j^n}{2}$$

$$(u_j^n)_t = \frac{u_j^{n+1} - u_j^n}{\tau}, \quad (u_j^n)_x = \frac{u_{j+1}^n - u_j^n}{h}, \quad (u_j^n)_{\bar{x}} = \frac{u_j^n - u_{j-1}^n}{h},$$

$$(u_j^n)_{\hat{x}} = \frac{u_{j+1}^n - u_{j-1}^n}{2h}, \quad (u^n, v^n)_h = h\left[\frac{1}{2}u_0^n\overline{v_0^n} + \sum_{j=1}^{J-1} u_j^n \overline{v_j^n} + \frac{1}{2}u_J^n\overline{v_J^n}\right],$$

$$\|u^n\|^2 = (u^n, u^n)_h, \quad \|u_x^n\|^2 = h\sum_{j=0}^{J-1} |(u_j^n)_x|^2, \quad \|u^n\|_\infty = \max_{0\le j\le J} |u_j^n|.$$

In this paper we denote C as a general positive constant which may have different values in different occurrences.

The scheme in [1] for the Kuramoto–Tsuzuki equation (1.1) is written as

$$(W_j^n)_t = (1 + \mathrm{i}c_1)(W_j^{n+\frac{1}{2}})_{x\bar{x}} + W_j^{n+\frac{1}{2}} - (1 + \mathrm{i}c_2)|W_j^{n+\frac{1}{2}}|^2 W_j^{n+\frac{1}{2}},$$
$$0 \le j \le J, \quad n = 0, 1, 2, \dots, N-1. \tag{2.1}$$

This scheme is a Crank–Nicolson one. The truncation error of this scheme is of order $O(h^2 + \tau^2)$. In order to obtain the solution W_j^{n+1} in the level $n + 1$, an outer nonlinear iteration for W_j^{n+1} needs to be done and the iterative values of W_j^{n+1} are solved by an inner linear system. Therefore, number of operators for the nonlinear implicit scheme is large.

In this paper, we propose the following difference scheme for the Kuramoto–Tsuzuki equation (1.1)

$$(W_j^n)_t = (1 + \mathrm{i}c_1)(W_j^{n+\frac{1}{2}})_{x\bar{x}} + W_j^{n+\frac{1}{2}} - (1 + \mathrm{i}c_2)\left(\frac{3}{2}|W_j^n|^2 - \frac{1}{2}|W_j^{n-1}|^2\right) W_j^{n+\frac{1}{2}},$$
$$0 \le j \le J, \quad n = 1, 2, \dots, N-1. \tag{2.2}$$

This scheme is an semi-explicit linearized Crank–Nicolson scheme. the truncation error of this scheme is of order $O(h^2+\tau^2)$. In this scheme, complicated nonlinear term $|W_j^{n+\frac{1}{2}}|^2$ is extrapolated by $\frac{3}{2}|W_j^n|^2 - \frac{1}{2}|W_j^{n-1}|^2$ in the scheme (2.1). Thus, we only need to solve a linear system of equations in computing W_j^{n+1}. Hence, the scheme (2.2) can be expected to be more efficient.

The initial and boundary conditions for the two schemes are

$$W_j^0 = w_0(x_j), \quad 1 \le j \le J-1, \tag{2.3}$$

$$(W_0^n)_{\hat{x}} = (W_J^n)_{\hat{x}} = 0, \quad n = 0, 1, 2, \dots, N. \tag{2.4}$$

It follows from (2.4) that

$$(W_j^n)_{x\bar{x}} = \begin{cases} \dfrac{2}{h^2}(W_1^n - W_0^n), & \text{when } j = 0, \\ \dfrac{2}{h^2}(W_{j+1}^n - 2W_j^n + W_{j-1}^n), & \text{when } 1 \le j \le J-1, \\ \dfrac{2}{h^2}(W_{J-1}^n - W_J^n), & \text{when } j = J. \end{cases} \tag{2.5}$$

The previous values W_j^n are used for the scheme (2.2) when new values W_j^{n+1} in the level $(n+1)$ are computed. Therefore, the scheme (2.2) cannot start by itself, the values W_j^1 in the scheme (2.2) should be computed by other algorithm in the starting step. It is clear that the scheme (2.1) may be applied to compute the values W_j^1. Then, the scheme (2.2) can be used to compute all values W_j^n for all $n \ge 2$.

3. Convergence and stability

In this section, we use an induction argument to prove the second-order convergence of the difference solution of the scheme (2.2)–(2.4).

First, some lemmas are introduced.

Lemma 3.1. *For any two discrete functions $\{u_j | j = 0, 1, \ldots, J\}$ and $\{v_j | j = 0, 1, \ldots, J\}$, there is the identity*

$$(u, v_{x\bar{x}})_h = -h \sum_{j=0}^{J-1} (u_j)_x (\overline{v_j})_x - u_0 (\overline{v_0})_{\hat{x}} + u_J (\overline{v_J})_{\hat{x}}. \tag{3.1}$$

Lemma 3.2 (*Sobolev's estimate [16]*). *For any discrete function $\{u_j^n | j = 0, 1, \ldots, J\}$ on the finite interval $[x_L, x_R]$, there is the inequality*

$$\|u^n\|_\infty \le C_0 \sqrt{\|u^n\|} \sqrt{\|u_x^n\| + \|u^n\|}, \tag{3.2}$$

where C_0 is a constant independent of $\{u_j^n | j = 0, 1, \ldots, J\}$ and step length h.

Lemma 3.3 (*Gronwall's inequality [16]*). *Suppose that the discrete function $\{w^n | n = 0, 1, 2, \ldots, N; N\tau = T\}$ satisfies the inequality*

$$w^n \le A + \tau \sum_{l=1}^{n} B_l w^l, \tag{3.3}$$

where A and $B_l (l = 0, 1, 2, \ldots, N)$ are nonnegative constants. Then

$$\max_{1 \le n \le N} |w^n| \le A e^{2\tau \sum_{l=1}^{N} B_l},$$

where τ is sufficiently small, such that $\tau \cdot (max_{1 \le l \le N} B_l) \le \frac{1}{2}$.

Lemma 3.3′ (*Gronwall's inequality [16]*). *Suppose that the discrete function $\{w^n | n = 0, 1, 2, \ldots, N; N\tau = T\}$ satisfies the inequality*

$$w^n - w^{n-1} \le A\tau w^n + B\tau w^{n-1} + C_n \tau, \tag{3.3′}$$

where A, B and $C_n (l = 0, 1, 2, \ldots, N)$ are nonnegative constants. Then

$$\max_{1 \le n \le N} |w^n| \le \left(w^0 + \tau \sum_{l=1}^{N} C_l \right) e^{2(A+B)T},$$

where τ is sufficiently small, such that $(A + B)\tau \le \frac{N-1}{2N}$, $(N > 1)$.

Lemma 3.4. *If $a \ge 0$, $b \ge 0$, there is the inequality*

$$ab \le \frac{a^p}{p} + \frac{b^q}{q},$$

or

$$ab \le \frac{(\varepsilon a)^p}{p} + \frac{(\frac{1}{\varepsilon} b)^q}{q},$$

where $p > 1$, $q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$.

The truncation error of the scheme (2.2)–(2.4) is written as follows

$$(w_j^n)_t - (1 + \mathrm{i}c_1)(w_j^{n+\frac{1}{2}})_{x\bar{x}} - w_j^{n+\frac{1}{2}} + (1 + \mathrm{i}c_2) \left(\frac{3}{2} |w_j^n|^2 - \frac{1}{2} |w_j^{n-1}|^2 \right) w_j^{n+\frac{1}{2}} = R_j^n,$$

$$0 \le j \le J, \quad n = 1, 2, \ldots, N - 1. \tag{3.4}$$

Using the similar method as the proof of Proposition 5.1 in [2] we obtain

Lemma 3.5. *Assume $u(x,t) \in C^{4,3}_{x,t}$, then the truncation error of scheme (2.2)–(2.4) satisfies*

$$|R^n_j| = O(\tau^2 + h^2).$$

Denote $e^n_j = w^n_j - W^n_j, j = 0, 1, 2, \ldots, J, n = 0, 1, 2, \ldots, N$, the following theorem on convergence of scheme (2.2)–(2.4) can be proved.

Theorem 3.1. *Suppose $w(x) \in C^{4,3}_{x,t}$, then the solution of the difference problem (2.2)–(2.4) converges to the solution of the problem (1.1) with order $O(h^2 + \tau^2)$ in the L_2-norm, if*

$$C_0 \cdot \max(C_{n-1}, C_n)\sqrt{\frac{2}{h} + 1}(h^2 + \tau^2) < 1, \tag{3.5}$$

and

$$\tau < \frac{1}{2\{3 + 6[C_w + 1]^2 + 12\sqrt{1 + c_2^2 C_w[2C_w + 1]}\}}. \tag{3.6}$$

Proof. From (3.1) we obtain

$$\begin{aligned}(w^n_j)_t \ &= \ (1 + \mathrm{i}c_1)(w_j^{n+\frac{1}{2}})_{x\bar{x}} + w_j^{n+\frac{1}{2}} - (1 + \mathrm{i}c_2)\left(\frac{3}{2}|w^n_j|^2 - \frac{1}{2}|w^{n-1}_j|^2\right) w_j^{n+\frac{1}{2}} + R^n_j, \\ &\quad 0 \le j \le J, \quad n = 1, 2, \ldots, N-1, \end{aligned} \tag{3.7}$$

$$w^0_j = w_0(x_j), \quad 0 \le j \le J. \tag{3.8}$$

Subtracting (2.2) and (2.3) from (3.7) and (3.8), respectively, we obtain the following error equations

$$\begin{aligned}(e^n_j)_t \ &= \ (1 + \mathrm{i}c_1)(e_j^{n+\frac{1}{2}})_{x\bar{x}} + e_j^{n+\frac{1}{2}} - (1 + \mathrm{i}c_2)\left(\frac{3}{2}|w^n_j|^2 - \frac{1}{2}|w^{n-1}_j|^2\right) w_j^{n+\frac{1}{2}} \\ &\quad + (1 + \mathrm{i}c_2)\left(\frac{3}{2}|W^n_j|^2 - \frac{1}{2}|W^{n-1}_j|^2\right) W_j^{n+\frac{1}{2}} + R^n_j, \quad 0 \le j \le J, \ n = 1, 2, \ldots, N-1, \end{aligned} \tag{3.9}$$

$$e^0_j = 0, \quad 0 \le j \le J. \tag{3.10}$$

Computing the inner product of (3.9) with $e^{n+\frac{1}{2}}$, then taking the real part of the result, we obtain

$$\begin{aligned}\frac{1}{2\tau}(\|e^{n+1}\|^2 - \|e^n\|^2) &= -\|e_x^{n+\frac{1}{2}}\|^2 + \|e^{n+\frac{1}{2}}\|^2 + \mathrm{Re}(R^n, e^{n+\frac{1}{2}})_h \\ &\quad - \mathrm{Re}\left\{(1 + \mathrm{i}c_2)\left(\left(\frac{3}{2}|w^n|^2 - \frac{1}{2}|w^{n-1}|^2\right) w^{n+\frac{1}{2}}\right.\right. \\ &\quad \left.\left. - \left(\frac{3}{2}|W^n|^2 - \frac{1}{2}|W^{n-1}|^2\right) W^{n+\frac{1}{2}}, e^{n+\frac{1}{2}}\right)_h\right\} \\ &= -\|e_x^{n+\frac{1}{2}}\|^2 + \|e^{n+\frac{1}{2}}\|^2 + \mathrm{Re}(R^n, e^{n+\frac{1}{2}})_h \\ &\quad - \mathrm{Re}\left\{(1 + \mathrm{i}c_2)\left(\left(\frac{3}{2}|w^n|^2 - \frac{3}{2}|W^n|^2 - \frac{1}{2}|w^{n-1}|^2 + \frac{1}{2}|W^{n-1}|^2\right) w^{n+\frac{1}{2}}, e^{n+\frac{1}{2}}\right)_h\right\} \\ &\quad - \mathrm{Re}\left\{\left(\left(\frac{3}{2}|W^n|^2 - \frac{1}{2}|W^{n-1}|^2\right) e^{n+\frac{1}{2}}, e^{n+\frac{1}{2}}\right)_h\right\} \quad n = 0, 1, 2, \ldots, N-1, \end{aligned} \tag{3.11}$$

where (2.4) and Lemma 3.1 are used.

Using the assumptions of the theorem and Lemma 3.4, we have

$$\|R^n\|_\infty \le C_R(h^2 + \tau^2), \quad \|w^n\|_\infty \le C_w, \quad 0 \le n\tau \le T, \tag{3.12}$$

where C_R and C_w are two constants independent of h and τ. It follows from the initial conditions that

$$\|e^0\| = 0, \quad \|W^0\|_\infty \le C_w. \tag{3.13}$$

Noting that the solution of the scheme (2.2) W^1 is computed by the scheme (2.1) in the first level of time. Hence, the following estimates are gotten in [2]:

$$\|e^1\| \le C_1(h^2 + \tau^2), \quad \|W^1\| \le C_w. \tag{3.14}$$

Now we use the induction argument to prove the convergence. Assume that

$$\|e^l\| \le C_l(h^2+\tau^2), \quad l=1,2,\ldots,n, \tag{3.15}$$

which leads to

$$\begin{aligned}\|e^l\|_\infty &\le C_0\sqrt{\|e^n\|}\sqrt{\|e^n_x\|+\|e^n\|} \le C_0\left(\sqrt{\frac{2}{h}+1}\right)\|e^l\| \\ &\le C_0C_l\sqrt{\frac{2}{h}+1}(h^2+\tau^2), \quad l=1,2,\ldots,n\end{aligned} \tag{3.16}$$

$$\|W^l\|_\infty \le \|W^n\|_\infty + \|e^n\|_\infty \le C_w + C_0C_l\sqrt{\frac{2}{h}+1}(h^2+\tau^2), \quad l=1,2,\ldots,n. \tag{3.17}$$

It follows from (3.11) that

$$\begin{aligned}\frac{1}{2\tau}(\|e^{n+1}\|^2-\|e^n\|^2) \le{}& \|e^{n+\frac12}\|^2+\frac12\|R^n\|^2+\frac12\|e^{n+\frac12}\|^2 \\ &+\sqrt{1+c_2^2}\left|\left(\left(\frac32|w^n|^2-\frac32|W^n|^2-\frac12|w^{n-1}|^2+\frac12|W^{n-1}|^2\right)w^{n+\frac12},e^{n+\frac12}\right)_h\right| \\ &+\left|\left(\left(\frac32|W^n|^2-\frac12|W^{n-1}|^2\right)e^{n+\frac12},e^{n+\frac12}\right)_h\right|, \quad n=1,2,\ldots,N-1,\end{aligned} \tag{3.18}$$

Since

$$\begin{aligned}&\left|\left(\left(\frac32|w^n|^2-\frac32|W^n|^2-\frac12|w^{n-1}|^2+\frac12|W^{n-1}|^2\right)w^{n+\frac12},e^{n+\frac12}\right)_h\right| \\ &=\Bigg|h\Bigg[\frac12\left(\frac32|w_0^n|^2-\frac32|W_0^n|^2-\frac12|w_0^{n-1}|^2+\frac12|W_0^{n-1}|^2\right)w_0^{n+\frac12}\overline{e_0^{n+\frac12}} \\ &\quad+\sum_{j=1}^{J-1}\left(\frac32|w_j^n|^2-\frac32|W_j^n|^2-\frac12|w_j^{n-1}|^2+\frac12|W_j^{n-1}|^2\right)w_j^{n+\frac12}\overline{e_j^{n+\frac12}} \\ &\quad+\frac12\left(\frac32|w_J^n|^2-\frac32|W_J^n|^2-\frac12|w_J^{n-1}|^2+\frac12|W_J^{n-1}|^2\right)w_J^{n+\frac12}\overline{e_J^{n+\frac12}}\Bigg]\Bigg| \\ &=\Bigg|h\Bigg[\frac12\left(\frac32e_0^n\overline{w_0^n}+\frac32W_0^n\overline{e_0^n}-\frac12e_0^{n-1}\overline{w_0^{n-1}}-W_0^{n-1}\overline{e_0^{n-1}}\right)w_0^{n+\frac12}\overline{e_0^{n+\frac12}} \\ &\quad+\sum_{j=1}^{J-1}\left(\frac32e_j^n\overline{w_j^n}+\frac32W_j^n\overline{e_j^n}-\frac12e_j^{n-1}\overline{w_j^{n-1}}-W_j^{n-1}\overline{e_j^{n-1}}\right)w_j^{n+\frac12}\overline{e_j^{n+\frac12}} \\ &\quad+\frac12\left(\frac32e_J^n\overline{w_J^n}+\frac32W_J^n\overline{e_J^n}-\frac12e_J^{n-1}\overline{w_J^{n-1}}-W_J^{n-1}\overline{e_J^{n-1}}\right)w_J^{n+\frac12}\overline{e_J^{n+\frac12}}\Bigg]\Bigg| \\ &\le\frac32C_w\left[2C_w+C_0\cdot\max(C_{n-1},C_n)\sqrt{\frac2h+1}(h^2+\tau^2)\right] \\ &\quad\times\left(\|e^n\|^2+\|e^{n+\frac12}\|^2+\|e^{n-1}\|^2\right), \quad n=1,2,\ldots,N-1,\end{aligned} \tag{3.19}$$

and

$$\begin{aligned}&\left|\left(\left(\frac32|W^n|^2-\frac12|W^{n-1}|^2\right)e^{n+\frac12},e^{n+\frac12}\right)_h\right| \le \left(\frac32\|W^n\|_\infty^2+\frac12\|W^{n-1}\|_\infty^2\right)\left(e^{n+\frac12},e^{n+\frac12}\right)_h \\ &\le 3\left[C_w+C_0\cdot\max(C_{n-1},C_n)\sqrt{\frac2h+1}(h^2+\tau^2)\right]^2\|e^{n+\frac12}\|^2, \quad n=1,2,\ldots,N-1,\end{aligned} \tag{3.20}$$

then it follows from (3.18) that

$$
\begin{aligned}
\frac{1}{2\tau}(\|e^{n+1}\|^2-\|e^n\|^2) \le\ & \|e^{n+\frac{1}{2}}\|^2+\frac{1}{2}\|R^n\|^2+\frac{1}{2}\|e^{n+\frac{1}{2}}\|^2\\
&+\frac{3}{2}\sqrt{1+c_2^2}C_w\left[2C_w+C_0\cdot\max(C_{n-1},C_n)\sqrt{\frac{2}{h}+1}(h^2+\tau^2)\right]\\
&\times(\|e^n\|^2+\|e^{n+\frac{1}{2}}\|^2+\|e^{n-1}\|^2)\\
&+3\left[C_w+C_0\cdot\max(C_{n-1},C_n)\sqrt{\frac{2}{h}+\frac{1}{p}}(h^2+\tau^2)\right]^2\|e^{n+\frac{1}{2}}\|^2\\
\le\ & \frac{1}{2}\|R^n\|^2+\frac{3}{4}(\|e^{n+1}\|^2+\|e^n\|^2)\\
&+\frac{3}{2}\sqrt{1+c_2^2}C_w\left[2C_w+C_0\cdot\max(C_{n-1},C_n)\sqrt{\frac{2}{h}+1}(h^2+\tau^2)\right]\\
&\times(\|e^{n+1}\|^2+2\|e^n\|^2+\|e^{n-1}\|^2)\\
&+\frac{3}{2}\left[C_w+C_0\cdot\max(C_{n-1},C_n)\sqrt{\frac{2}{h}+1}(h^2+\tau^2)\right]^2(\|e^{n+1}\|^2+\|e^n\|^2),\\
&n=1,2,\ldots,N-1,
\end{aligned}
\tag{3.21}
$$

which leads to

$$
\begin{aligned}
\|e^{n+1}\|^2 \le\ & \|e^n\|^2+\tau\Bigg\{\|R^n\|^2+\frac{3}{2}(\|e^{n+1}\|^2+\|e^n\|^2)\\
&+3\sqrt{1+c_2^2}C_w\left[2C_w+C_0\cdot\max(C_{n-1},C_n)\sqrt{\frac{2}{h}+1}(h^2+\tau^2)\right](\|e^{n+1}\|^2+2\|e^n\|^2+\|e^{n-1}\|^2)\\
&+3\left[C_w+C_0\cdot\max(C_{n-1},C_n)\sqrt{\frac{2}{h}+1}(h^2+\tau^2)\right]^2(\|e^{n+1}\|^2+\|e^n\|^2)\Bigg\}\\
\le\ & \|e^1\|^2+\tau\sum_{l=1}^{n}\|R^l\|^2+\sum_{l=0}^{n+1}\Bigg\{3+6\left[C_w+C_0(\max_{0\le l\le n}C_l)\sqrt{\frac{2}{h}+1}(h^2+\tau^2)\right]^2\\
&+12\sqrt{1+c_2^2}C_w\left[2C_w+C_0\cdot\max(C_{n-1},C_n)\sqrt{\frac{2}{h}+1}(h^2+\tau^2)\right]\Bigg\}\|e^l\|^2.
\end{aligned}
\tag{3.22}
$$

Taking small h and τ such that

$$
C_0\cdot\max(C_{n-1},C_n)\sqrt{\frac{2}{h}+1}(h^2+\tau^2)<1. \tag{3.23}
$$

Then, using Lemma 3.3 and (3.22), for

$$
\tau<\frac{1}{2\{3+6[C_w+1]^2+12\sqrt{1+c_2^2}C_w[2C_w+1]\}}, \tag{3.24}
$$

we have

$$
\|e^{n+1}\|^2\le(T(C_R)^2+C_1^2)(h^2+\tau^2)^2\mathrm{e}^{2T\left\{3+6[C_w+1]^2+12\sqrt{1+c_2^2}C_w[2C_w+1]\right\}}\le(C_{n+1})^2(h^2+\tau^2)^2, \tag{3.25}
$$

and

$$
C_{n+1}=(\sqrt{T}C_R+C_1)\mathrm{e}^{T\left\{3+6[C_w+1]^2+12\sqrt{1+c_2^2}C_w[2C_w+1]\right\}}\equiv C_c,
$$

where C_c is a constant independent of n. $\square$

Lemma 3.6. *Suppose $u(x,t) \in C^{4,3}_{x,t}$, if τ and h are small enough, then for $n = 0, 1, 2, \dots, N$, the following inequality*

$$\|W^n\|_\infty \le \widetilde{C}_0 \tag{3.26}$$

hold, where $\widetilde{C}_0$ is a constant independent of h and τ.

Proof. It follows from Theorem 3.1 that, for sufficiently small τ and h,

$$\|W^n\|_\infty \le \|w^n\|_\infty + \|e^n\|_\infty \le \widetilde{C}_0. \tag{3.27}$$

This completes the proof of Lemma 3.6. □

Theorem 3.2. *Under the conditions of Theorem 3.1, the solution of the difference problem (2.2)–(2.4) converges to the solution of the problem (1.1) with order $O(h^2 + \tau^2)$ in the L_∞-norm.*

Proof. Computing the inner product of (3.9) with $e_t^n/(1 + \mathrm{i}c_1)$, we obtain

$$\begin{aligned}\frac{1}{1+\mathrm{i}c_1}\|e_t^n\|^2 &= -h\sum_{j=0}^{J-1}\left(e_j^{n+\frac12}\right)_x(\overline{e_j^n})_{xt} + \frac{1}{1+\mathrm{i}c_1}\left(e^{n+\frac12}, e_t^n\right)_h + \frac{1}{1+\mathrm{i}c_1}(R^n, e_t^n)_h \\ &\quad - \frac{1+\mathrm{i}c_2}{1+\mathrm{i}c_1}\left(\left(\frac32|w^n|^2 - \frac12|w^{n-1}|^2\right)w^{n+\frac12} - \left(\frac32|W^n|^2 - \frac12|W^{n-1}|^2\right)W^{n+\frac12}, e_t^n\right)_h \\ &\quad n = 0, 1, 2, \dots, N-1,\end{aligned} \tag{3.28}$$

where Lemma 3.1 is used. Using (3.27), we obtain

$$\begin{aligned}&\left(\frac32|w_j^n|^2 - \frac12|w_j^{n-1}|^2\right)w_j^{n+\frac12} - \left(\frac32|W_j^n|^2 - \frac12|W_j^{n-1}|^2\right)W_j^{n+\frac12} \\ &\le \left(\frac32|w_j^n|^2 + \frac12|w_j^{n-1}|^2 + \frac32|W_j^n|^2 + \frac12|W_j^{n-1}|^2\right)\left(w_j^{n+\frac12} - W_j^{n+\frac12}\right) \\ &\le 2(C_w + \widetilde{C}_0)^2|e_j^{n+\frac12}|.\end{aligned} \tag{3.29}$$

Thus, taking the real part of (3.28), we obtain

$$\begin{aligned}\frac{1}{1+c_1^2}\|e_t^n\|^2 &= -\frac{1}{2\tau}(\|e_x^{n+1}\|^2 - \|e_x^n\|^2) + \mathrm{Re}\left\{\frac{1}{1+\mathrm{i}c_1}(e^{n+\frac12}, e_t^n)_h\right\} + \mathrm{Re}\left\{\frac{1}{1+\mathrm{i}c_1}(R^n, e_t^n)_h\right\} \\ &\quad - \mathrm{Re}\left\{\frac{1+\mathrm{i}c_2}{1+\mathrm{i}c_1}\left(\left(\frac32|w^n|^2 - \frac12|w^{n-1}|^2\right)w^{n+\frac12} - \left(\frac32|W^n|^2 - \frac12|W^{n-1}|^2\right)W^{n+\frac12}, e_t^n\right)_h\right\} \\ &\le -\frac{1}{2\tau}(\|e_x^{n+1}\|^2 - \|e_x^n\|^2) + \frac{1}{\sqrt{1+c_1^2}}\|e^{n+\frac12}\|\cdot\|e_t^n\| + \frac{1}{\sqrt{1+c_1^2}}\|R^n\|\cdot\|e_t^n\| \\ &\quad - 2\frac{\sqrt{1+c_2^2}}{\sqrt{1+c_1^2}}(C_w + \widetilde{C}_0)^2\|e^{n+\frac12}\|\cdot\|e_t^n\| \\ &\le -\frac{1}{2\tau}(\|e_x^{n+1}\|^2 - \|e_x^n\|^2) + \frac{1}{4(1+c_1^2)}\|e_t^n\|^2 + \|e^{n+\frac12}\|^2 + \frac{1}{2(1+c_1^2)}\|e_t^n\|^2 + \frac12\|R^n\|^2 \\ &\quad \times \frac{1}{4(1+c_1^2)}\|e_t^n\|^2 + 4(1+c_2^2)(C_w + \widetilde{C}_0)^4\|e^{n+\frac12}\|^2, \quad n = 1, 2, \dots, N-1,\end{aligned} \tag{3.30}$$

which leads to

$$\|e_x^{n+1}\|^2 - \|e_x^n\|^2 \le \tau\|R^n\|^2 + \tau[1 + 4(1+c_2^2)(C_w + \widetilde{C}_0)^4](\|e_x^{n+1}\|^2 + \|e_x^n\|^2), \quad n = 1, 2, \dots, N-1. \tag{3.31}$$

It follows from (3.31) and Lemma 3.3′ that

$$\max_{1\le n\le N}\|e_x^n\|^2 \le \left(\|e_x^1\|^2 + \tau\sum_{l=1}^{N}\|R^n\|^2\right)\mathrm{e}^{4[1+4(1+c_2^2)(C_w+\widetilde{C}_0)^4]T}, \tag{3.32}$$

where τ is sufficiently small, such that $2[1+4(1+c_2^2)(C_w+\widetilde{C_0})^4]\tau \le \frac{N-1}{2N}$, $(N>1)$ It follows from the proof of theorem 6.1 in [2] that there exists a positive constant $\widetilde{C_1}$ independent of h and τ such that

$$\|e_x^1\|^2 \le \widetilde{C_1}(h^2+\tau^2). \tag{3.33}$$

Thus, it follows from (3.32) and (3.33) that

$$\begin{aligned}\max_{1\le n\le N}\|e_x^n\|^2 &\le (\widetilde{C_1}^2+TC_R^2)(h^2+\tau^2)^2 e^{4[1+4(1+c_2^2)(C_w+\widetilde{C_0})^4]T}\\ &\le C_T^2(h^2+\tau^2)^2,\end{aligned} \tag{3.34}$$

where $C_T=(\widetilde{C_1}+\sqrt{T}C_R)e^{2[1+4(1+c_2^2)(C_w+\widetilde{C_0})^4]T}$. $\square$

By the similar proof of Theorem 3.2, we can obtain the following theorem.

Theorem 3.3. *Under the conditions of Theorem 3.1, the scheme (2.2)–(2.4) is stable in the L_2-norm for the initial values.*

Another similar semi-explicit scheme for the GL equation (1.1) is written as

$$\begin{aligned}&(W_j^n)_t=(1+ic_1)\left(W_j^{n+\frac12}\right)_{x\bar{x}}+W_j^{n+\frac12}-(1+ic_2)\left(\left|\frac32 W_j^n-\frac12 W_j^{n-1}\right|^2\right)W_j^{n+\frac12},\\ &\quad 0\le j\le J,\quad n=1,2,\ldots,N-1.\end{aligned} \tag{3.35}$$

Its second-order convergence in the uniform norm and stability can be proved by the similar proof as that of the scheme (1.2) in [2].

4. Inhomogeneous equation

For inhomogeneous equation

$$\frac{\partial w}{\partial t}=(1+ic_1)\frac{\partial^2 w}{\partial x^2}+w-(1+ic_2)|w|^2w+f(x,t),\quad (x,t)\in(0,1)\times(0,T], \tag{4.1}$$

$$w(x,0)=w_0(x),\quad x\in[0,1], \tag{4.2}$$

$$\frac{\partial w}{\partial x}(0,t)=0,\quad \frac{\partial w}{\partial x}(1,t)=0,\quad t\in(0,T], \tag{4.3}$$

where $f(x,t)$ is a known complex valued smooth function. One revises (2.2)–(2.4) and (3.35), (2.3)–(2.4) as follows

$$\begin{aligned}(W_j^n)_t &= (1+ic_1)(W_j^{n+\frac12})_{x\bar{x}}+W_j^{n+\frac12}-(1+ic_2)\left(\frac32|W_j^n|^2-\frac12|W_j^{n-1}|^2\right)W_j^{n+\frac12}+F_j^{k+\frac12},\\ &\qquad 0\le j\le J,\quad n=1,2,\ldots,N-1.\end{aligned} \tag{4.4}$$

$$W_j^0=w_0(x_j),\quad 1\le j\le J-1, \tag{4.5}$$

$$(W_0^n)_{\hat{x}}=(W_J^n)_{\hat{x}}=0,\quad n=0,1,2,\ldots,N, \tag{4.6}$$

and

$$\begin{aligned}(W_j^n)_t &= (1+ic_1)\left(W_j^{n+\frac12}\right)_{x\bar{x}}+W_j^{n+\frac12}-(1+ic_2)\left(\left|\frac32 W_j^n-\frac12 W_j^{n-1}\right|^2\right)W_j^{n+\frac12}+F_j^{n+\frac12},\\ &\qquad 0\le j\le J,\quad n=1,2,\ldots,N-1.\end{aligned} \tag{4.7}$$

$$W_j^0=w_0(x_j),\quad 1\le j\le J-1, \tag{4.8}$$

$$(W_0^n)_{\hat{x}}=(W_J^n)_{\hat{x}}=0,\quad n=0,1,2,\ldots,N, \tag{4.9}$$

where

$$F_j^{n+\frac12}=\begin{cases} f\left(\frac{h}{3},t_{n+\frac12}\right), & \text{when } j=0,\\ f(x_j,t_{n+\frac12}), & \text{when } 1\le j\le J-1,\\ f\left(1-\frac{h}{3},t_{n+\frac12}\right), & \text{when } j=J.\end{cases} \tag{4.10}$$

and $t_{n+\frac12}=(t_n+t_{n+1})/2$. The truncation errors cab be proved by the method in [3] to be $O(h^2+\tau^2)$. And it is not difficult to prove that both of the two schemes are stable and convergent at the rate of second order in L_∞-norm.

Table 1
$\|e^N\|_\infty$ computed by the difference scheme (4.4)–(4.6) for solving Example 1 at $T = 2$.

$h = \tau$	$\lambda = \tau/h^2$	$\|e^N\|_\infty$	$\|e^N(h,\tau)\|_\infty / \|e^{2N}\left(\frac{h}{2},\frac{\tau}{2}\right)\|_\infty$	$\|e^N\|_\infty/(h^2+\tau^2)$
0.1000	10	3.0629E−2	–	1.5314
0.0500	20	7.5197E−3	4.0731	1.5040
0.0250	40	1.8714E−3	4.0184	1.4971
0.0125	80	4.6730E−4	4.0046	1.4954

Table 2
$\|e^N\|_\infty$ computed by the difference scheme (4.7)–(4.9) for solving Example 1 at $T = 2$.

$h = \tau$	$\lambda = \tau/h^2$	$\|e^N\|_\infty$	$\|e^N(h,\tau)\|_\infty / \|e^{2N}\left(\frac{h}{2},\frac{\tau}{2}\right)\|_\infty$	$\|e^N\|_\infty/(h^2+\tau^2)$
0.1000	10	4.2749E−2	–	2.1374
0.0500	20	1.0422E−2	4.1019	2.0844
0.0250	40	2.5990E−3	4.0100	2.0791
0.0125	80	6.5058E−4	3.9949	2.0818

Table 3
Comparison of the difference schemes for solving Example 1 with $h = \tau = 0.005$ at $T = 10$.

Schemes	$\|e^N\|_\infty$	CPU (s)	$\|e^N\|_\infty/(h^2+\tau^2)$
Scheme 1	8.0793E−4	203	16.1587
Scheme 2	1.2681E−3	89	25.3632
Scheme 3	3.7577E−3	78	75.1547
Scheme 4	9.4701E−4	83	18.9403
Scheme 5	1.4241E−3	83	28.4822

Remark 4.1. Above results also hold for periodic boundary-initial value problem of the Kuramoto–Tsuzuki equation.

5. Numerical experiment

In this section, we compute two examples to verify the correction of our theoretical analysis in above sections. For convenience, we denote the scheme (9.2) and (9.3) in [2], the scheme in [3] and the our two new schemes (2.2) and (3.35) as Schemes 1–5, respectively.

Example 1 (*[2]*). Consider the inhomogeneous equation

$$\begin{aligned}\frac{\partial w}{\partial t} &= (1+2\mathrm{i})\frac{\partial^2 w}{\partial x^2} + w - (1-3\mathrm{i})|w|^2 w \\ &\quad + [\pi^2 - \sin^2 \pi x + \mathrm{i}(2t + 2\pi^2 - 3\cos^2 \pi x)]e^{\mathrm{i}t^2} \cos \pi x, \quad (x,t) \in (0,1)\times(0,T], \end{aligned} \tag{5.1}$$

$$w(x,0) = \cos \pi x, \quad x \in [0,1], \tag{5.2}$$

$$\frac{\partial w}{\partial x}(0,t) = 0, \quad \frac{\partial w}{\partial x}(1,t) = 0, \quad t \in (0,T], \tag{5.3}$$

whose exact solution is $w(x,t) = \mathrm{e}^{\mathrm{i}t^2}\cos \pi x$. In the following tables, $\|e^N\|_\infty = \max_{0\le j\le J} |W_j^N - w_j^N|$, $\|e^N(h,\tau)\|_\infty$ denotes the error $\|e^N\|_\infty$ with the step h and τ.

Tables 1 and 2 show the second-order convergence in L_∞-norm of the two schemes, which verifies the correction of the theoretical analysis. One can see that the two schemes are very robust even for large λ, which is why they are be called the robust schemes.

Table 3 gives the accuracy and CPU time of the five schemes for solving Example 1, one see that Scheme 4 is the best one of them not only for accuracy but also for CPU time in implementation.

Using the extrapolation technique in implementation of the scheme (4.7)–(4.9) and the scheme (4.4)–(4.6) for solving Example 1, one can obtain high accuracy. Let $W_j^n(h,\tau)$ denote the numerical solution W_j^n with the steps h and τ at the point $(jh, n\tau)$ and $W_{2j}^{2n}(h/2,\tau/2)$ denote the numerical solution W_{2j}^{2n} with the steps $h/2$ and $\tau/2$ at the point $((2j)h/2, (2n)\tau/2)$. We denote

$$\|\widetilde{e}^N(h,\tau)\|_\infty = \max_{0\le j\le J}\left|\frac{4}{3}W_{2j}^{2n}(h/2,\tau/2) - \frac{1}{3}W_j^n(h,\tau) - w(jh,n\tau)\right|.$$

Tables 4 and 5 show the high accuracy of the scheme (4.4)–(4.6) and the scheme (4.7)–(4.9) when using the extrapolation technique, and the order of accuracy is $O(h^4+\tau^4)$.

Table 4
$\|\widetilde{e}^N\|_\infty$ computed by the difference scheme (4.4)–(4.6) for solving Example 1 at $T = 2$.

h	$\|e^N\|_\infty$	$\|\widetilde{e}^N(h,\tau)\|_\infty / \|\widetilde{e}^{2N}\left(\frac{h}{2},\frac{\tau}{2}\right)\|_\infty$	$\|\widetilde{e}^N\|_\infty/(h^4+\tau^4)$
0.1000	1.8385E−4	–	0.9193
0.0500	1.1488E−5	16.0033	0.9191
0.0250	7.1362E−7	16.0985	0.9134

Table 5
$\|\widetilde{e}^N\|_\infty$ computed by the difference scheme (4.7)–(4.9) for solving Example 1 at $T = 2$.

h	$\|e^N\|_\infty$	$\|\widetilde{e}^N(h,\tau)\|_\infty / \|\widetilde{e}^{2N}\left(\frac{h}{4},\frac{\tau}{4}\right)\|_\infty$	$\|\widetilde{e}^N\|_\infty/(h^4+\tau^4)$
0.1000	3.7035E−4	–	1.8518
0.0250	1.4309E−6	258.8105	1.8317

Table 6
Comparison of the accuracy of the five schemes using extrapolation technique for solving Example 1 with $h = \tau = 0.05$ at $T = 10$.

Schemes	$\|\widetilde{e}^N(h,\tau)\|_\infty$	$\|\widetilde{e}^N(h/2,\tau/2)\|_\infty$
Scheme 1	2.9079E−2	1.3897E−3
Scheme 2	NAN	9.0280E−3
Scheme 3	7.5530E−1	4.3053E−2
Scheme 4	4.3572E−2	1.9805E−3
Scheme 5	7.9755E−3	3.8501E−4

Table 7
$\|e^N\|_\infty$ computed by the difference scheme (2.2) for solving Example 2 at $T = 5$.

h	τ	$\lambda = \tau/h^2$	$\|e^N\|_\infty$	$\|e^N(h,\tau)\|_\infty / \|e^{2N}\left(\frac{h}{2},\frac{\tau}{2}\right)\|_\infty$	$\|e^N\|_\infty/(h^2+\tau^2)$
$\pi/10$	0.1000	1.0132	3.8393E−3	–	3.5321E−2
$\pi/20$	0.0500	2.0264	9.6976E−4	3.9590	3.5687E−2
$\pi/40$	0.0250	4.0528	2.4362E−4	3.9806	3.5861E−2
$\pi/80$	0.0125	8.1057	6.1050E−5	3.9905	3.5946E−2

Table 8
$\|e^N\|_\infty$ computed by the difference scheme (3.35) for solving Example 2 at $T = 5$.

h	τ	$\lambda = \tau/h^2$	$\|e^N\|_\infty$	$\|e^N(h,\tau)\|_\infty / \|e^{2N}\left(\frac{h}{2},\frac{\tau}{2}\right)\|_\infty$	$\|e^N\|_\infty/(h^2+\tau^2)$
$\pi/10$	0.1000	1.0132	2.9852E−3	–	2.7464E−2
$\pi/20$	0.0500	2.0264	7.5220E−4	3.9687	2.7681E−2
$\pi/40$	0.0250	4.0528	1.8854E−4	3.9895	2.7753E−2
$\pi/80$	0.0125	8.1057	4.7182E−5	3.9961	2.7781E−2

Table 6 shows, when the extrapolation technique is used in implementation, Scheme 4 is still more accurate than the other two linear schemes in [2,3] and as accurate as the Crank–Nicolson scheme which is a nonlinear one. A surprise result found out in the numerical example is, when the extrapolation technique used for solving the difference schemes, Scheme 5 is the most accurate one when $h \in [0.01, 0.05]$ and $\tau = h$, such as what Table 6 shows.

Example 2. Consider the inhomogeneous equation

$$\frac{\partial w}{\partial t} = (1+\mathrm{i})\frac{\partial^2 w}{\partial x^2} + w - (1+\mathrm{i})|w|^2 w, \quad (x,t) \in \mathcal{R} \times (0,T], \tag{5.4}$$

$$w(x,0) = \frac{\sqrt{3}}{2}\mathrm{e}^{\mathrm{i}\frac{x}{2}}, \quad x \in \mathcal{R}, \tag{5.5}$$

$$w(x,t) = w(x+4\pi,t) \quad t \in (0,T], \tag{5.6}$$

whose exact solution is $w(x,t) = \frac{\sqrt{3}}{2}\mathrm{e}^{\mathrm{i}\left(\frac{x}{2}-t\right)}$.

Tables 7 and 8 also show the second-order convergence in L_∞-norm of the scheme (2.2) and the scheme (3.35) for solving periodic boundary-initial value problem, which verifies the correction of the theoretical analysis again. It is seen from Tables 7 and 8 that the two schemes are also robust even for large λ for solving the periodic boundary-initial value problem.

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